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The North Sea Problem VI

Non-stationary wind-effects in a rectangular bay

Theoretical Part

by

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List of symbols

x, y	Cartesian coordinates. The bay is determined in dimensionless coordinates by $0 < x < \pi$, $0 < y < b$;
t	the time;
u, v	the components of the total stream;
φ	the elevation of the surface;
U, V	the components of the surface stress due to the windfield;
Ω	the coefficient of Coriolis;
λ	a coefficient of friction;
h	the depth;
g	the constant of gravity;
c	velocity of propagation of free waves, $c^2 = gh$;
p	the variable of the Laplace transformation;
κ	defined by $\kappa^2 = p(p+\lambda) + \Omega^2 p(p+\lambda)^{-1}$;
$\mathrm{tg}\,\gamma$	defined by $\mathrm{tg}\,\gamma = \Omega(p+\lambda)^{-1}$;
q	defined by $q^2 = p(p+\lambda)$;
r	defined by $r^2 = p\,\Omega$;
s	defined by $s^2 = \Omega^2 p(p+\lambda)^{-1}$;
θ	defined by $\theta = \mathrm{cotg}\,\gamma$;
ν_n	defined for $n=1,2,3\ldots$ by $\nu_n^2 = n^2+\kappa^2$;
θ_n	defined for $n=1,2,3\ldots$ by $\theta_n = \theta n \nu_n^{-1}$;
α_n	defined for $n=1,2,3\ldots$ by $\alpha_n = r^2 n^{-1} \nu_n^{-1}$;
α	defined by $\alpha = 1 - 2\gamma/\pi$.

1. Introduction

In this paper we consider a mathematical model for the behaviour of the North Sea under a storm. In this model the North Sea is represented by a rectangular bay which is bounded on three sides by coasts and which at the fourth side borders on an ocean. The depth is assumed to be uniform. For the sake of simplicity the longest axis is assumed to coincide with the North-South direction. The Southern border then corresponds with i.a. the Dutch coast and the middle of it roughly with the position of Den Helder. The influence of a storm is expressed through the stress it exerts on the surface of the sea. This stress (U,V) may be dependent onthe coordinates (x,y) and the time t.

The mathematical problem is formulated in section 2 as a set of linear partial differential equations (2.1) with boundary conditions (2.2) for the components of the total stream (u,v) and the elevation ζ. After removal of the time variable by means of a Laplace transformation (2.8) elimination of $\bar{u}$ and $\bar{v}$, the bar indicating the Laplace transform, leads to an elliptic partial differential equation of the Helmholtz type (2.11) for $\bar{\zeta}$. The determination of $\bar{\zeta}$ follows in two steps. The first step consists in solving the problem for the strip $0 < x < \pi$ i.e. without paying attention to the boundary conditions at the coast y=0 and the ocean y=b. If $\bar{\zeta}_0$ represent such a solution the difference $\bar{\zeta} - \bar{\zeta}_0$ may be considered as a free motion in the strip $0 < x < \pi$. It is possible to determine this free motion which consists of a linear superposition of two Kelvin waves and $2 \times \infty$ Poincaré waves in such a way that the coast condition at y=0 and the ocean condition at y=b are satisfied.

The solution $\bar{\zeta}_0$ of the strip problem requires the solution of the corresponding problem of Green. The latter problem is solved in section 3. In order to avoid overloading the analysis in the remainder of the paper the special case of a uniform Northern wind is considered. Yet this particular case which is considered in section 4 shows all peculiarities of the general problem. However, in that case $\bar{\zeta}_0$ can be found without taking recourse to the Green's problem. The explicit expression of $\bar{\zeta}$ (4.12) involves a set of coefficients A_m and a set B_m (m=0,1,2...), where A_0 and B_0 belong to the Kelvin waves, A_m (m=1,2,3...) to the Poincaré waves at the coast y=0 and B_m (m=1,2,3...) to the Poincaré waves at the ocean. Assuming b to be sufficiently large the Poincaré waves at the coast y=0 and those at the ocean at y=b do not interact. Therefore the coast condition which is considered in section 5 gives a set of linear relations between A_m (m=0,1,2...) and B_0 only.

In a similar way the ocean condition which is considered in section 6 gives a set of linear relations between B_m $(m=0,1,2...)$ and A_o. It is now possible to determine the coefficients by means of some iterative process.

Some information concerning the analytical dependence of A_m and B_n on p, λ and Ω can be obtained by considering the limit cases $\Omega \rightarrow 0$ and $\Omega \rightarrow \infty$. If Ω is small we may consider $\bar{\zeta}$ to be dependent on the parameter groups q and r only. Expansion of A_m, B_m in rising powers of r^2 eventually leads to the approximation (7.1) of section 7. In this approximation only the first-order correction is given. This correction vanishes at the North-South axis and shows that in a first approximation the rotation of the Earth gives rise to a skew-symmetric obliqueness of the sea-level. If Ω is large $\bar{\zeta}$ may be considered to depend on s and θ only, s being moderate and θ small. In a similar way series expansion in powers of θ results in certain approximations of the coefficients A_m, B_m which are to be found in the sections 5, 6 and 8. The first-order approximation of $\bar{\zeta}$ at the South-coast takes the simple form (8.6). We note that in this approximation $\bar{\bar{\zeta}}$ is uniform along the South-coast. A second-order approximation is given by (8.5), (8.10) and (8.13). Then a slight skew-symmetric dependence on x becomes apparent.

In applying the results obtained above to the North Sea we note that the assumption of a large value of b certainly is correct. However, the value of Ω is rather large and the results pertaining to small Ω (or r^2) must be considered as being of theoretical interest only. On the other hand the approximation (8.6) derived for small θ and moderate s appears to be quite good which has been confirmed by numerical work.

Numerical applications will be considered in the subsequent paper.

2. The mathematical problem

The problem can be described mathematically by the equations of motion (I 2.6) and the equation of continuity (I 2.7)

$$
(2.1)\qquad \begin{cases} (\frac{\partial}{\partial t} + \lambda)u - \Omega v + c^2 \frac{\partial \zeta}{\partial x} = U \\ (\frac{\partial}{\partial t} + \lambda)v + \Omega u + c^2 \frac{\partial \zeta}{\partial y} = V \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \end{cases}
$$

and the boundary conditions

$$
(2.2)\qquad \begin{cases} u = 0 & \text{for} \quad x = 0 \quad \text{and} \quad x = a \\ v = 0 & \text{for} \quad y = 0 \\ \zeta = 0 & \text{for} \quad y = b \;. \end{cases}
$$

The components U and V of the wind force are related to the velocity of the wind at sealevel v_s in the following way (see II 2.4)

$$
(2.3)\qquad \sqrt{U^2+V^2} = 3.0 \times 10^{-6}\, v_s^2 \;.
$$

For the problem considered here dimensionless quantities will be introduced according to

$$
(2.4)\qquad \begin{cases} (x,y) \longrightarrow \pi^{-1}a\,(x,y) & t \rightarrow \pi^{-1}a\,c^{-1}\,t \\ (u,v) \longrightarrow h\,c\,\,(u,v) & \zeta \longrightarrow h\zeta \\ (U,V) \longrightarrow \pi a^{-1} h c^2\,(U,V). \end{cases}
$$

This has the effect that the constants c and a occurring in (2.1) and (2.2) can be formally replaced by c=1 and a=π respectively.

For the numerical application to the North Sea case the following numerical values will be chosen (see II section 2)

$$
(2.5)\qquad \begin{cases} a = 400 \text{ km} & \Omega = 0.43 \text{ h}^{-1} \\ b = 800 \text{ km} & \lambda = 0.2\,\Omega \\ h = 65 \text{ m} & c = 91 \text{ km/h} \;. \end{cases}
$$

If these values are substituted in (2.4) the dimensionless time scale becomes 1.4 hour. The dimensionless values of Ω, λ and b are approximately

$$
(2.6)\qquad \begin{cases} \Omega = 0.6 & \lambda = 0.12 \\ b = 2\pi \;. \end{cases}
$$

The relation (2.3) is in dimensionless form

$$\sqrt{U^2+V^2} = 0.93 \times 10^{-5} \, v_s^{\,2} . \tag{2.7}$$

A general discussion of the problem (2.1) (2.2) has been given in I section 2. In order to avoid continuous reference to the first paper of this series the most important formulae will be repeated here. The Laplace transformation

$$\bar{\varphi}(x,y,p) \stackrel{\text{def}}{=} \int_0^\infty e^{-pt} \varphi(x,y,t)dt \tag{2.8}$$

changes (2.1) and (2.2) in

$$\begin{cases} (p+\lambda)\bar{u} - \Omega\bar{v} + \dfrac{\partial\bar{\varphi}}{\partial x} = \bar{U} \\ (p+\lambda)\bar{v} + \Omega\bar{u} + \dfrac{\partial\bar{\varphi}}{\partial y} = \bar{V} \\ \dfrac{\partial\bar{u}}{\partial x} + \dfrac{\partial\bar{v}}{\partial y} + p\bar{\varphi} = 0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \bar{u} = 0 & \text{for } x = 0 \text{ and } x = \pi \\ \bar{v} = 0 & \text{for } y = 0 \\ \bar{\varphi} = 0 & \text{for } y = b. \end{cases} \tag{2.10}$$

Elimination of $\bar{u}$ and $\bar{v}$ gives for $\bar{\varphi}$ the non-homogeneous Helmholtz equation

$$(\Delta - \kappa^2)\bar{\varphi} = \bar{F}(x,y,p), \tag{2.11}$$

with

$$\bar{F}(x,y,p) = \left(\frac{\partial\bar{U}}{\partial x} + \frac{\partial\bar{V}}{\partial y}\right) + \operatorname{tg}\gamma\left(\frac{\partial\bar{V}}{\partial x} - \frac{\partial\bar{U}}{\partial y}\right), \tag{2.12}$$

and where

$$\kappa^2 \stackrel{\text{def}}{=} p(p+\lambda) + \Omega^2 p(p+\lambda)^{-1}, \tag{2.13}$$

and

$$\operatorname{tg}\gamma \stackrel{\text{def}}{=} \Omega(p+\lambda)^{-1}. \tag{2.14}$$

From the equations (2.9) it can be derived that

$$\begin{cases} p^{-1}\kappa^2\bar{u} = -\dfrac{\partial\bar{\varphi}}{\partial x} - \operatorname{tg}\gamma\,\dfrac{\partial\bar{\varphi}}{\partial y} + \bar{U} + \operatorname{tg}\gamma\,\bar{V} \\ p^{-1}\kappa^2\bar{v} = -\dfrac{\partial\bar{\varphi}}{\partial y} + \operatorname{tg}\gamma\,\dfrac{\partial\bar{\varphi}}{\partial x} + \bar{V} - \operatorname{tg}\gamma\,\bar{U} . \end{cases} \tag{2.15}$$

By means of these expressions the boundary conditions at $x=0$, $x=\pi$ and $y=0$ can be expressed in terms of $\bar{\varphi}$. By using (2.15) they can be written as

$$\frac{\partial\bar{\varphi}}{\partial x} + \operatorname{tg}\gamma\,\frac{\partial\bar{\varphi}}{\partial y} = \bar{U} + \operatorname{tg}\gamma\,\bar{V} \qquad \text{for } x=0 \text{ and } x=\pi, \tag{2.16}$$

$$\frac{\partial\bar{\varphi}}{\partial y} - \operatorname{tg}\gamma\,\frac{\partial\bar{\varphi}}{\partial x} = \bar{V} - \operatorname{tg}\gamma\,\bar{U} \qquad \text{for } y=0, \tag{2.17}$$

$$\bar{\varphi} = 0 \qquad \text{for } y=b. \tag{2.18}$$

From the formulation of the problem by (2.11), (2.13) and (2.14) it follows that $\bar{\varphi}$ depends essentially on the two independent parameter combinations κ^2 and $\operatorname{tg}\gamma$. The same is true for $p^{-1}\bar{u}$ and $p^{-1}\bar{v}$ in view of the relations (2.15). Of course κ^2 and $\operatorname{tg}\gamma$ may be replaced by any other pair of independent combinations of p, λ and Ω. We shall consider two limit cases which may be characterized by either Ω small or Ω large. In the first case it is convenient to use the parameters

$$q^2 \overset{\text{def}}{=} p(p+\lambda) \qquad r^2 \overset{\text{def}}{=} p\,\Omega\,. \tag{2.19}$$

In fact we have

$$q = \kappa\cos\gamma \qquad r = \kappa\sqrt{\sin\gamma\,\cos\gamma}\,. \tag{2.20}$$

In the second case we shall use the groups

$$s^2 \overset{\text{def}}{=} \Omega^2 p(p+\lambda)^{-1} \qquad \theta \overset{\text{def}}{=} (p+\lambda)\Omega^{-1}. \tag{2.21}$$

In fact we have

$$s = \kappa\sin\gamma \qquad \theta = \operatorname{cotg}\gamma\,. \tag{2.22}$$

For an arbitrary windfield (U,V) the solution of (2.11), (2.16), (2.17) and (2.18) may be reduced to that of the corresponding problem of Green. However, it has not been found possible to solve the latter problem in an explicit way. Yet an explicit function of Green can be derived for an infinite strip $0 < x < \pi$, $-\infty < y < \infty$ with homogeneous boundary conditions of the type (2.10). This problem will be studied in the following section. Then by means of the function of Green for a strip the solution of the original problem for the bay $0 < x < \pi$, $0 < y < b$ may be reduced eventually to that of solving a double set of an infinite number of linear equations. We shall write

$$\bar{\varphi}(x,y,p) = \bar{\varphi}_0(x,y,p) + \bar{\varphi}_1(x,y,p), \tag{2.23}$$

where $\bar{\varphi}_0(x,y,p)$ is a particular solution of (2.11) which satisfies the boundary conditions (2.16) but not necessarily those at y=0 and y=b. Such a solution can be deduced from the function of Green for the strip $0 < x < \pi$, $-\infty < y < \infty$. Consequently $\bar{\varphi}_1(x,y,p)$ satisfies a <u>homogeneous</u> equation of Helmholtz and <u>homogeneous</u> boundary conditions at $x=0$ and $x=\pi$. Hence this function can be composed of the Kelvin- and Poincaré-waves belonging to the infinite strip $0 < x < \pi$, $-\infty < y < \infty$. (See IV section 3). If A_o and B_o are the coefficients of the two Kelvin waves and A_j and B_j $(j=1,2,3...)$ those of the Poincaré-waves the boundary conditions (2.16) and (2.17) each eventually lead to an infinite number of linear equations. The problem of determining the coefficients is quite formidable. However, if b is large (see IV section 7) these equations reduce to the simple form

$$\sum_{n=0}^{\infty} \alpha_{mn} A_n = \alpha_m B_o + O(e^{-b}) \tag{2.24}$$

and

$$\sum_{n=0}^{\infty} \beta_{mn} B_n = \beta_m A_o + O(e^{-b}), \tag{2.25}$$

for $m=0,1,2...$, and where the order terms represent the contribution of the terms which vanish for $b \to \infty$. The coefficients α_{mn}, α_m, β_{mn}, β_m are only used here and have no relation to similar symbols elsewhere in this paper.

3. Problem of Green

The problem of Green for the strip $0 < x < \pi$, $-\infty < y < \infty$ mentioned in the preceding section may be formulated as follows

(3.1) $$(\Delta - \kappa^2)G(x,y,\xi,\eta,\gamma) = -\delta(x-\xi)\,\delta(y-\eta),$$

with the boundary conditions

(3.2) $$\frac{\partial G}{\partial x} + \mathrm{tg}\,\gamma\frac{\partial G}{\partial y} = 0 \qquad \text{for } x=0 \text{ and } x=\pi .$$

Further it will be required that for $|y| \to \infty$ the function of Green is of the order $\exp - \varepsilon|y|$, where ε is an arbitrarily small positive quantity.

From Green's theorem it can easily be derived that

(3.3) $$G(x,y,\xi,\eta,\gamma) = G(\xi,\eta,x,y,-\gamma).$$

When G is known the function $\bar{\varphi}_0(x,y)$ which satisfies (2.11) and (2.15) follows by means of the same theorem

(3.4) $$\bar{\varphi}_0(x,y) = -\int_{\infty}^{\infty}\int_0^{\pi} G(x,y,\xi,\eta,\gamma)\bar{f}(\xi,\eta)\,d\xi\,d\eta +$$
$$- \int_{\infty}^{\infty} G(x,y,0,\eta,\gamma)\bar{f}(0,\eta)\,d\eta + \int_{-\infty}^{\infty} G(x,y,\pi,\eta,\gamma)\bar{f}(\pi,\eta)\,d\eta$$

where

(3.5) $$\bar{f}(x,y) = \bar{U}(x,y) + \mathrm{tg}\,\gamma\,\bar{V}(x,y).$$

Substitution of (2.12), (3.5) and partial integration gives

(3.6) $$\bar{\varphi}_0(x,y) = \int_{-\infty}^{\infty}\int_0^{\pi}\left\{\bar{U}\left(\frac{\partial}{\partial\xi} - \mathrm{tg}\,\gamma\,\frac{\partial}{\partial\eta}\right)G + \bar{V}\left(\frac{\partial}{\partial\eta} + \mathrm{tg}\,\gamma\,\frac{\partial}{\partial\xi}\right)G\right\}d\xi\,d\eta .$$

The explicit form of G will not be derived here systematically but in the following simple albeit somewhat heuristic way.
The free waves of the strip are the Kelvin waves and the Poincaré waves (see IV section 3). The Kelvin waves are

(3.7) $$\exp \pm (sx - qy).$$

The Poincaré waves are for n=1,2,3...

(3.8) $$\begin{cases} (\sin nx + \theta_n \cos nx)e^{-\nu_n y} \\ (\sin nx - \theta_n \cos nx)e^{\nu_n y} \end{cases},$$

where

(3.9) $$\nu_n \overset{\text{def}}{=} \sqrt{n^2+\kappa^2} ,$$

and

$$\theta_n \overset{\text{def}}{=} \frac{n}{\nu_n}\theta . \tag{3.10}$$

In view of the symmetry relation (3.3) we shall try to represent G in the following way

$$G = C\, e^{s(x+\xi-\pi)-q(y-\eta)} + \tag{3.11}$$

$$+\sum_{n=1}^{\infty} C_n(\sin nx+\theta_n\cos nx)(\sin n\xi+\theta_n\cos n\xi)e^{-\nu_n(y-\eta)}$$

for $y > \eta$ and

$$G = C\, e^{-s(x+\xi-\pi)+q(y-\eta)} + \tag{3.12}$$

$$+\sum_{n=1}^{\infty} C_n(\sin nx-\theta_n\cos nx)(\sin n\xi-\theta_n\cos n\xi)e^{-\nu_n(\eta-y)}$$

for $y < \eta$.

First continuity at $y=\eta$ requires that

$$C\,\mathrm{sh}\, s(x+\xi-\pi) + \sum_{n=1}^{\infty}\theta_n C_n \sin n(x+\xi)=0. \tag{3.13}$$

Substitution of (3.11) and (3.12) in (3.1) gives, using differentiation in a generalized sense,

$$(\Delta-\kappa^2)G = \left\{\frac{\partial}{\partial y}G(y>\eta) - \frac{\partial}{\partial y}G(y<\eta)\right\}\delta(y-\eta) = \tag{3.14}$$

$$= \Big\{-2qC\,\mathrm{ch}\, s(x+\xi-\pi) +$$

$$-2\sum_{n=1}^{\infty}\nu_n C_n(\sin nx\sin n\xi + \theta_n^2\cos nx\cos n\xi)\Big\}\delta(y-\eta).$$

Since by differentiation of (3.13) it follows that

$$2qC\,\mathrm{ch}\, s(x+\xi-\pi) + 2\sum_{n=1}^{\infty}\nu_n\theta_n^2 C_n\cos n(x+\xi) = 0 \tag{3.15}$$

the right-hand side of (3.14) reduces to

$$\Big\{-2\kappa^2 s^{-2}\sum_{n=1}^{\infty}(n^2+s^2)\nu_n^{-1}C_n\sin nx\sin n\xi\Big\}\delta(y-\eta). \tag{3.16}$$

Therefore we must have that for $0 < x, \xi < \pi$

$$-2\kappa^2 s^{-2}\sum_{n=1}^{\infty}(n^2+s^2)\nu_n^{-1}C_n\sin nx\sin n\xi = \delta(x-\xi). \tag{3.17}$$

It is possible to satisfy the relations (3.13) and (3.17) for the same coefficients. We have

$$C_n = \frac{s^2}{\pi\kappa^2}\,\frac{\nu_n}{n^2+s^2}, \tag{3.18}$$

and

(3.19) $$C = \frac{r^2}{2\kappa^2 \text{sh } s\pi}.$$

Substitution of (3.18) and (3.19) in (3.11) and (3.12) gives the following explicit expression

(3.20) $$G(x,y,\xi,\eta,\gamma) = \frac{r^2}{2\kappa^2 \text{sh } s\pi} e^{\sigma\{(x+\xi-\pi)-q(y-\eta)\}} +$$

$$+ \frac{s^2}{\pi\kappa^2} \sum_{n=1}^{\infty} \frac{\nu_n}{n^2+s^2} (\sin nx + \sigma\theta_n \cos nx)(\sin n\xi + \sigma\theta_n \cos n\xi) e^{-\nu_n |y-\eta|},$$

where

(3.21) $$\sigma = \text{sgn}(y-\eta).$$

4. Solution for a uniform windfield

For a uniform windfield the solution $\bar{\varphi}_o$ of (2.23) can be obtained in a direct way without using the Green's function which has been derived in the previous section. In particular we shall consider the case of a uniform "northern" wind

$$U = 0 \qquad V = V(t). \tag{4.1}$$

Without loss of generality we may take

$$V(t) = -\delta(t) \tag{4.2}$$

i.e. a momentary "Northern" disturbance. Hence we have $\bar{V} = -1$.

According to (2.11) and (2.15) the problem can be reformulated by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \kappa^2\right)\bar{\varphi} = 0, \tag{4.3}$$

$$\kappa^2 \frac{\bar{u}}{p} = -\frac{\partial\bar{\varphi}}{\partial x} - \operatorname{tg}\gamma \frac{\partial\bar{\varphi}}{\partial y} - \operatorname{tg}\gamma , \tag{4.4}$$

$$\kappa^2 \frac{\bar{v}}{p} = -\frac{\partial\bar{\varphi}}{\partial y} + \operatorname{tg}\gamma \frac{\partial\bar{\varphi}}{\partial x} - 1 , \tag{4.5}$$

with the boundary conditions

$$\bar{u} = 0 \qquad \text{for} \quad x = 0 \ \text{and} \ x = \pi , \tag{4.6}$$
$$\bar{v} = 0 \qquad \text{for} \quad y = 0 , \tag{4.7}$$
$$\bar{\varphi} = 0 \qquad \text{for} \quad y = b . \tag{4.8}$$

Supposing that $\bar{u}_o, \bar{v}_o$ and $\bar{\varphi}_o$ depend on x only it is easily found that

$$\frac{\bar{u}_o}{p} = -\frac{\operatorname{tg}\gamma}{\kappa^2}\left\{1 - \frac{\operatorname{ch}\kappa(\frac{1}{2}\pi - x)}{\operatorname{ch}\frac{1}{2}\kappa\pi}\right\}, \tag{4.9}$$

$$\frac{\bar{v}_o}{p} = -\frac{1}{\kappa^2}\left\{1 + \operatorname{tg}^2\gamma \frac{\operatorname{ch}\kappa(\frac{1}{2}\pi - x)}{\operatorname{ch}\frac{1}{2}\kappa\pi}\right\}, \tag{4.10}$$

and

$$\bar{\varphi}_o = \operatorname{tg}\gamma \frac{\operatorname{sh}\kappa(\frac{1}{2}\pi - x)}{\kappa \operatorname{ch}\frac{1}{2}\kappa\pi} . \tag{4.11}$$

The elevation φ_1 represents a free motion in the channel $0 < x < \pi$ and may be considered to be a linear superposition of the two Kelvin waves (3.7) and the double infinity of Poincaré waves (3.10). Following (2.23) we put

$$\begin{aligned}\bar{\varphi}(x,y,p) = {} & \bar{\varphi}_o(x,p) + A_o \operatorname{sh}\{s(x-\tfrac{1}{2}\pi)-qy\} + B_o \operatorname{ch}\{s(x-\tfrac{1}{2}\pi)-qy\} + \\ & + \sum_{n=1}^{\infty} n^{-1}A_n(\sin nx+\theta_n\cos nx)e^{-\nu_n y} + \\ & + \sum_{n=1}^{\infty} n^{-1}B_n(\sin nx-\theta_n\cos nx)e^{-\nu_n(b-y)} ,\end{aligned} \tag{4.12}$$

where q and s are given by (2.19) and (2.21) and where θ_n is determined by (3.10) and (3.9).

The components of the total stream of the Kelvin waves and the system of Poincaré waves are given in the following table where α_n is defined by

(4.13) $$\alpha_n \overset{def}{=} \frac{r^2}{n\nu_n} \qquad n=1,2,3...$$

and where r^2 is given by (2.19).

	Kelvin waves	Poincaré waves, n=1,2,3...
$\bar{u}/p$	0	$\pm(n^2+s^2)\sin nx \exp \mp \nu_n y$
$\bar{v}/p$	$\pm \exp \pm \{s(x-\frac{1}{2}\pi)-qy\}$	$n\nu_n(\cos nx \pm \alpha_n \sin nx)\exp \mp \nu_n y$
$\bar{\varphi}$	$q \exp \pm \{s(x-\frac{1}{2}\pi)-qy\}$	$r^2\nu_n(\sin nx \pm \theta_n \cos nx)\exp \mp \nu_n y$

table 4.1

Then for the components of the total stream we have

(4.14) $$\frac{1}{p}\bar{u}(x,y,p) = \frac{1}{p}\bar{u}_o(x,p) + \frac{1}{r^2}\sum_{n=1}^{\infty}\frac{n^2+s^2}{n\nu_n}A_n \sin nx\, e^{-\nu_n y} + $$
$$- \frac{1}{r^2}\sum_{n=1}^{\infty}\frac{n^2+s^2}{n\nu_n}B_n \sin nx\, e^{-\nu_n(b-y)}$$

and

(4.15) $$\frac{1}{p}\bar{v}(x,y,p) = \frac{1}{p}\bar{v}_o(x,p) + \frac{1}{q}A_o \mathrm{ch}\{s(x-\tfrac{1}{2}\pi)-qy\} +$$
$$+ \frac{1}{q}B_o \mathrm{sh}\{s(x-\tfrac{1}{2}\pi)-qy\} +$$
$$+ \frac{1}{r^2}\sum_{n=1}^{\infty}A_n(\cos nx+\alpha_n \sin nx)e^{-\nu_n y} + \frac{1}{r^2}\sum_{n=1}^{\infty}B_n(\cos nx-\alpha_n \sin nx)e^{-\nu_n(b-y)}.$$

The coast condition (4.7) gives by substituting y=0 in (4.15)

(4.16) $$s A_o \mathrm{ch}\, s(\tfrac{1}{2}\pi-x) - s B_o \mathrm{sh}\, s(\tfrac{1}{2}\pi-x) + \sum_{n=1}^{\infty}A_n(\cos nx+\alpha_n \sin nx) =$$
$$= \frac{r^2}{\kappa^2}\left\{1 + \mathrm{tg}^2\gamma\, \frac{\mathrm{ch}\,\kappa(\frac{1}{2}\pi-x)}{\mathrm{ch}\,\frac{1}{2}\kappa\pi}\right\} + O(e^{-b}),$$

where the order term represents the contribution of the Poincaré waves at the ocean boundary. If b is sufficiently large this contribution can be neglected. The condition (4.16) will be discussed in section 5.

The ocean condition (4.8) gives by substituting y=b in (4.12)

(4.17) $$-A_o \mathrm{sh}\{s(\tfrac{1}{2}\pi-x)+qb\} + B_o \mathrm{ch}\{s(\tfrac{1}{2}\pi-x)+qb\} + \sum_{n=1}^{\infty}n^{-1}B_n(\sin nx-\theta_n \cos nx) =$$
$$= -\mathrm{tg}\,\gamma\, \frac{\mathrm{sh}\,\kappa(\frac{1}{2}\pi-x)}{\kappa\, \mathrm{ch}\,\frac{1}{2}\kappa\pi} + O(e^{-b}),$$

where in a similar way the order term represents the contribution of the Poincaré waves at the coast y=0. The condition (4.17) will be discussed in section 6.

5. Coast condition

From the coast condition (4.16) a set of linear equations can be derived by expanding both sides of (4.16) in a pure cosine series and equating corresponding coefficients. For convenience we shall introduce the following notations

(5.1) $$s_m = \frac{2}{\pi}\int_0^\pi \cos mx \; e^{s(\frac{1}{2}\pi - x)} dx,$$

(5.2) $$\Gamma_{mn} = \frac{2}{\pi}\int_0^\pi \cos mx \sin nx \; dx,$$

(5.3) $$v_m = \frac{2}{\pi\kappa^2}\int_0^\pi \cos mx \left\{1 + tg^2\gamma \frac{ch\, \kappa(\frac{1}{2}\pi - x)}{ch\, \frac{1}{2}\kappa\pi}\right\} dx.$$

Performing the integrations we find

(5.4) $$\begin{cases} m \text{ even} & s_m = \frac{4s}{\pi}\, sh\, \frac{1}{2} s\pi/(m^2+s^2) \\ m \text{ odd} & s_m = \frac{4s}{\pi}\, ch\, \frac{1}{2} s\pi/(m^2+s^2). \end{cases}$$

$$\begin{matrix} m+n \text{ even} & \Gamma_{mn} = 0 \\ m+n \text{ odd} & \Gamma_{mn} = \frac{4}{\pi}\frac{n}{n^2-m^2}. \end{matrix}$$

(5.6) $$\begin{cases} & v_o = \frac{2}{\kappa^2}\left\{1 + tg^2\gamma \frac{th\, \frac{1}{2}\kappa\pi}{\frac{1}{2}\kappa\pi}\right\} \\ m \text{ even} & v_m = \frac{2\, tg^2\gamma}{m^2+\kappa^2}\, \frac{th \frac{1}{2}\kappa\pi}{\frac{1}{2}\kappa\pi} \\ m \text{ odd} & v_m = 0. \end{cases}$$

Then the result of the cosine expansion of both sides of (4.16) is with omission of the order term

(5.7) $$\begin{cases} & s_o\, s\, A_o = r^2 v_o - r^2 \Sigma_1 \frac{\Gamma_{on}}{n\nu_n} A_n \\ m \text{ even} & A_m = -s_m s\, A_o + r^2 v_m - r^2 \Sigma_1 \frac{\Gamma_{mn}}{n\nu_n} A_n \\ m \text{ odd} & A_m = s_m s\, B_o - r^2 \Sigma_2 \frac{\Gamma_{mn}}{n\nu_n} A_n, \end{cases}$$

where Σ_1 denotes a summation over odd indices n=1,3,5... and Σ_2 one over even indices n=2,4,6... . This system which is of the form (2.24) can be solved by means of an iterative process. The ultimate result may be written in the form

(5.8) $$A_m = \emptyset_m(B_o) + O(e^{-b}),\; m=1,2,3...$$

It is obvious that $A_m = O(m^{-2})$ for $m \to \infty$ since the A_m are the coefficients of a cosine expansion.

We shall consider the system (5.7) in the two cases which are characterized by either Ω small or Ω large. In the first case we shall take q and r as independent parameters. Then the coefficients A_m may be considered as a power series in r^2. We note that

$$(5.9)\quad \begin{cases} s = r^2/q \\ v_o = 2/q^2 + O(r^4) \\ v_m = O(r^4) \end{cases} \qquad \begin{aligned} s_o &= 2 + O(r^4) \\ s_m &= O(r^4) \text{ for m even} \\ s_m &= \frac{4s}{m^2\pi} + O(r^6) \text{ for m odd.} \end{aligned}$$

If we assume that $B_o = O(1)$, which will be confirmed in the next section, where the ocean condition is considered, it follows from (5.7) that

$$(5.10)\quad \begin{cases} A_o = q^{-1} + O(r^4) \\ A_m = \dfrac{4s^2}{\pi m^2} B_o + O(r^8) & \text{for m odd} \\ A_m = O(r^6) & \text{for m even.} \end{cases}$$

In the second case s and θ will be chosen as the independent parameters. Then the coefficients A_m may be considered as power series in θ. We note that

$$(5.11)\quad \begin{cases} r^2 = \theta\, s^2 \qquad\qquad \kappa^2 = (1+\theta^2)s^2 \\ v_m = \dfrac{2}{\theta^2(s^2+m^2)} \dfrac{\text{th}\,\frac{1}{2}s\pi}{\frac{1}{2}s\pi} + O(1) \quad \text{for } m=0,2,4\ldots . \end{cases}$$

If we assume that $B_o = O(\theta^{-1})$ which again is confirmed later on it follows from (5.7) that

$$(5.12)\quad \begin{cases} A_o = \dfrac{1}{\theta s\, \text{ch}\,\frac{1}{2}s\pi} + O(1) \\ A_m = s_m\, s\, B_o + O(\theta) & \text{for m odd} \\ A_m = O(1) & \text{for m even.} \end{cases}$$

6. Ocean condition

The ocean condition (4.17) calls for a more extensive treatment. This relation will be written as

(6.1) $$\sum_{n=1}^{\infty} n^{-1}B_n(\sin nx-\theta_n\cos nx) = \varphi(x),$$

where

(6.2) $$\varphi(x) = A_o \mathrm{sh}\left\{s(\tfrac{1}{2}\pi -x)+qb\right\} - B_o \mathrm{ch}\left\{s(\tfrac{1}{2}\pi-x)+qb\right\} - \mathrm{tg}\,\gamma \frac{\mathrm{sh}\,\kappa(\frac{1}{2}\pi-x)}{\kappa\,\mathrm{ch}\,\frac{1}{2}\kappa\pi}$$

Expansions of this kind have been studied elsewhere 1). A summary of the results will be given below. We note that for $n \to \infty$

(6.3) $$\theta_n = \theta + O(n^{-2}).$$

Therefore the properties of the expansion (6.1) are very similar to those of the simpler expansion

(6.4) $$\sum_{n=1}^{\infty} b_n(\sin nx - \theta\cos nx) = \beta(x)$$

in the interval $0 < x < \pi$.

The expansion (6.4) is unique and the asymptotic behaviour of b_n is of the following kind

(6.5) $$b_n = B\,n^{-1+\alpha} + (-1)^{n-1} B' n^{-1-\alpha} + B'' n^{-3+\alpha} + \ldots$$

where

(6.6) $$\alpha \overset{\mathrm{def}}{=} \frac{2}{\pi}\,\mathrm{arctg}\,\theta = 1 - \frac{2\gamma}{\pi}.$$

The explicit expression of B is

(6.7) $$B = -\frac{2^{\alpha}\cos\frac{1}{2}\alpha\pi}{\pi\,\Gamma(\alpha)} \int_0^{\pi} \mathrm{tg}^{\alpha-1}\tfrac{1}{2}x\ \beta(x)dx.$$

According to (6.5) the convergence of (6.4) is of subharmonic order unless $B = 0$ when it is of hyperharmonic order.
To the set $\sin nx - \theta\cos nx$, $n=1,2,3\ldots$ a biorthogonal set of functions is associated, each being the product of the factor $\mathrm{tg}^{\alpha-1}\frac{1}{2}x$ and a finite trigonometric sum. We note that

(6.8) $$\int_0^{\pi} (\sin nx-\theta\cos nx)\mathrm{tg}^{\alpha-1}\tfrac{1}{2}x\,dx = 0 \qquad \text{for } n=1,2,3\ldots$$

and that

(6.9) $$\frac{\sin\frac{1}{2}\alpha\pi}{\pi}\int_0^{\pi} \mathrm{tg}^{\alpha-1}\tfrac{1}{2}x\,dx = 1.$$

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1) Cf. Lauwerier [2].

The properties of the expansion (6.1) can easily be derived from those of the expansion (6.4). In particular we have for $n \to \infty$

(6.10) $$B_n = B\, n^{\alpha} + (-1)^{n-1} B' n^{-\alpha} + B'' n^{-2+\alpha} + \ldots$$

where the constants on the right-hand side are not the same of course as those of (6.5). There exists also an orthogonal function $h_o(x)$ with the properties

(6.11) $$\int_0^{\pi} (\sin nx - \theta_n \cos nx) h_o(x) dx = 0 \quad \text{for} \quad n=1,2,3\ldots$$

and

(6.12) $$\frac{1}{\pi} \int_0^{\pi} h_o(x) dx = 1.$$

Once the asymptotic behaviour of the coefficients B_n is known something can be said concerning the behaviour of $\bar{\varphi}$ and its partial derivatives at the corners $(0,b)$ and (π,b) of the rectangular basin.

If at $(0,b)$ local polar coordinates (ρ, φ) are introduced by means of

(6.13) $$x = \rho \cos \varphi \qquad\qquad y = b - \rho \sin \varphi$$

it follows from (4.12) and (6.10) after some elementary reductions that for $\rho \to 0$

(6.14) $$\bar{\varphi} = B \sec \tfrac{1}{2}\alpha\pi \;\; \operatorname{Im}\left\{ \sum_{n=1}^{\infty} n^{-1+\alpha} \exp i(n \rho e^{i\varphi} - \tfrac{1}{2}\alpha\pi) \right\} + o(1),$$

and next

(6.15) $$\bar{\varphi} = B\, \Gamma(\alpha) \sec \tfrac{1}{2}\alpha\pi \;\; \rho^{-\alpha} \;\; \sin \alpha\varphi + o(1).$$

Hence a solution where $\bar{\varphi}$ is continuous at $(0,b)$ requires that $B=0$. It appears that then also u and v are continuous at this point. A similar analysis shows that at (π,b), $\bar{\varphi}$ is continuous for $B=0$. However, the partial derivatives of $\bar{\varphi}$ at the latter point are infinite.

From (6.1) a set of linear relations of the type

(6.16) $$\sum_{n=1}^{\infty} c_{mn} B_n = c_m B_o + d_m A_o + e_m$$

can be derived in a similar way as in the previous section. A necessary and sufficient condition for $B=0$, where B is the coefficient of n^{α} in the asymptotic expansion of B, is

(6.17) $$\int_0^{\pi} h_o(x)\, \varphi(x) dx = 0.$$

Substitution of (6.2) yields a linear relation between A_o and B_o only. Further linear relations may be obtained e.g. by expanding each side of (6.1) in a sine series and equating corresponding coefficients.

The first orthogonal function $h_o(x)$ may be approximated by

$$h_o(x) \approx \sin \tfrac{1}{2}\alpha\pi \ \ tg^{\alpha-1} \tfrac{1}{2}x. \tag{6.18}$$

We may also try to find an expansion of $h_o(x)$ for the two special cases of the preceding section.

In the first case where q and r are the independent variables we have

$$\theta_n = \frac{nq^2}{\nu_n} \frac{1}{r^2}, \qquad \kappa = q + O(r^4). \tag{6.19}$$

Putting tentatively

$$h_o(x) = 1 + r^2 \psi(x) + O(r^4) \tag{6.20}$$

it follows from (6.11) and (6.12) that

$$\int_0^{\pi} \{1 + r^2 \psi(x) + \ldots\} \{ \cos nx - \frac{\nu_n}{nq^2} r^2 \sin nx \} dx = 0 \tag{6.21}$$

and

$$\int_0^{\pi} \psi(x) dx = 0. \tag{6.22}$$

From (6.21) we obtain at once

$$\int_0^{\pi} \cos nx \ \ \psi(x) dx = \frac{\nu_n}{nq^2} \int_0^{\pi} \sin nx \ dx \tag{6.23}$$

so that in view of (6.22)

$$\psi(x) = \frac{2}{\pi q^2} \sum_1 \frac{\nu_n}{n^2} \cos nx. \tag{6.24}$$

If now (6.20) and (6.2) are substituted in (6.17) it follows after some simple reductions that

$$B_o = \text{th } qb \ A_o \{ 1 + O(r^4) \}, \tag{6.25}$$

so that with the value of A_o as given in (5.10) we find

$$B_o = \frac{\text{th } qb}{q} \{ 1 + O(r^4) \}. \tag{6.26}$$

For the coefficient B_m a similar treatment is possible. In that case the (m+1)th biorthogonal function $h_m(x)$ associated to $(\cos nx - \theta_n^{-1} \sin nx)$ n=1,2,3... may be expanded in a similar way as (6.20) by

$$h_m(x) = \cos mx + r^2 \psi_m(x) + O(r^4). \tag{6.27}$$

Without giving details we mention the following results

$$\psi_m(x) = \frac{1}{q^2} \sum_{n=1}^{\infty} \frac{\nu_n}{n} \Gamma_{mn} \cos nx, \tag{6.28}$$

$$(6.29)\begin{cases} B_m = \dfrac{4\nu_m r^4}{\pi m^2 q^4}\left(-\dfrac{1}{\operatorname{ch} qb} + \dfrac{m^2}{m^2+q^2}\right) + O(r^8) & \text{for } m \text{ odd} \\ B_m = O(r^6) & \text{for m even.} \end{cases}$$

In the second case where s and θ are the independent variables the ocean condition (6.1) can be written in the form

$$(6.30) \qquad \sum_{n=1}^{\infty} n^{-1}B_n(\sin nx - \theta \frac{n}{\nu_n}\cos nx) = A_o \operatorname{sh}\left\{s(\tfrac{1}{2}\pi - x)+qb\right\} + \\ -B_o \operatorname{ch}\left\{s(\tfrac{1}{2}\pi - x)+qb\right\} - \frac{1}{\theta}\frac{\operatorname{sh} s(\frac{1}{2}\pi - x)}{s \operatorname{ch} \frac{1}{2}s\pi} + O(\theta).$$

For the first orthogonal function $h_o(x)$ we may put in view of (6.8) and (6.9)

$$(6.31) \qquad h_o(x) = \pi\delta(x) + \theta\,\chi(x) + O(\theta^2).$$

The condition

$$(6.32) \qquad \int_0^{\pi}\left\{\pi\delta(x) + \theta\,\chi(x)+\ldots\right\}\left\{\sin nx - \theta\frac{n}{\nu_n}\cos nx\right\}dx = 0$$

for all $n \geq 1$ leads in a similar way as above to

$$(6.33) \qquad \chi(x) = 2\sum_{n=1}^{\infty}\frac{n}{\nu_n}\sin nx,$$

where the right-hand side may be interpreted as a generalized function. However, we may also write

$$(6.33^a) \qquad \chi(x) = \operatorname{cotg}\tfrac{1}{2}x - 2\sum_{n=1}^{\infty}(1-\frac{n}{\nu_n})\sin nx,$$

where now the series on the right-hand side of (6.33) converges in ordinary sense.

In the lowest order approximation the relation (6.17) reduces to

$$(6.34) \qquad \int_0^{\pi}\delta(x)\left[A_o \operatorname{sh}\left\{s(\tfrac{1}{2}\pi - x)+qb\right\} - B_o \operatorname{ch}\left\{s(\tfrac{1}{2}\pi - x)+qb\right\} + \\ - \frac{\operatorname{sh} s(\frac{1}{2}\pi - x)}{\theta\, s \operatorname{ch}\frac{1}{2}s\pi}\right]dx = 0,$$

from which we obtain

$$(6.35) \qquad A_o \operatorname{sh}(\tfrac{1}{2}s\pi + qb) - B_o \operatorname{ch}(\tfrac{1}{2}s\pi + qb) = \frac{\operatorname{th}\frac{1}{2}s\pi}{\theta s} + O(1).$$

Substitution of the lowest order approximation of A_o from (5.12) gives at once the fundamental result

$$(6.36) \qquad B_o = \frac{\operatorname{sh}(\frac{1}{2}s\pi + qb) - \operatorname{sh}\frac{1}{2}s\pi}{\theta s \operatorname{ch}(\frac{1}{2}s\pi + qb)\operatorname{ch}\frac{1}{2}s\pi} + O(1).$$

The lowest order approximation of the coefficients B_n can be derived as follows. Substitution of the approximations of A_o and B_o from (5.12) and (6.36) in $\varphi(x)$ of (6.2) gives

(6.37) $$\varphi(x) = \frac{\text{ch } qb - 1}{\theta s \text{ ch}(\frac{1}{2}s\pi + qb)\text{ch}\frac{1}{2}s\pi} \text{ sh } sx + O(1).$$

Then from the lowest order approximation of (6.1) viz.

(6.38) $$\sum_{n=1}^{\infty} n^{-1} B_n \sin nx = \varphi(x) + O(1)$$

it is easily obtained that

(6.39) $$B_n = \frac{4(\text{ch } qb-1)\text{sh}\frac{1}{2}s\pi}{\pi\theta s \text{ ch}(\frac{1}{2}s\pi + qb)} \frac{(-1)^{n-1}n^2}{n^2+s^2} + O(1).$$

A second order approximation will be discussed in section 8.

7. Approximation for small Ω

In this case q and r are chosen as the independent parameters. Substitution of the approximations (5.10), (6.26) and (6.29) in (4.12) gives

$$\bar{\varphi}(x,y) = \frac{\operatorname{sh} q(b-y)}{q \operatorname{ch} qb} + \frac{r^2}{q^2}\left\{\frac{\operatorname{sh} q(\frac{1}{2}\pi - x)}{q \operatorname{ch} \frac{1}{2} q\pi} - \frac{(\frac{1}{2}\pi - x)\operatorname{ch} q(b-y)}{\operatorname{ch} qb} + \right.$$

$$(7.1) \qquad + \frac{4}{\pi} q \operatorname{th} qb \sum_1 \frac{\cos nx \exp -y\sqrt{n^2+q^2}}{n^2\sqrt{n^2+q^2}} +$$

$$\left. - \frac{4}{\pi}\sum_1\left(\frac{n^2}{n^2+q^2} - \frac{1}{\operatorname{ch} qb}\right)\frac{\cos nx \exp -(b-y)\sqrt{n^2+q^2}}{n^2}\right\} +$$

$$+ O(r^4) + O(e^{-b}).$$

We note that for $\Omega \longrightarrow \infty$

$$(7.2) \qquad \bar{\varphi}(x,y) \longrightarrow \frac{\operatorname{sh} q(b-y)}{q \operatorname{ch} qb}.$$

For the components of the total stream it follows then from (4.4) and (4.5) that

$$(7.3) \qquad \bar{u}(x,y) \longrightarrow 0 ,$$

$$(7.4) \qquad p^{-1}\bar{v}(x,y) \longrightarrow \frac{1}{q^2}\left\{1 - \frac{\operatorname{ch} q(b-y)}{\operatorname{ch} qb}\right\}.$$

As a specialization of (7.1) we note that at the axis $x=\frac{1}{2}\pi$

$$(7.5) \qquad \bar{\varphi}(\tfrac{1}{2}\pi, y) = \frac{\operatorname{sh} q(b-y)}{q \operatorname{ch} qb} + O(r^4) + O(e^{-b}),$$

so that the influence of the coefficient of Coriolis Ω upon the elevation at the axis is of higher order than elsewhere. Hence the rotation of the Earth gives rise in a first instance to a skew-symmetric obliqueness of the sea-level. Or in formula form

$$(7.6) \qquad \bar{\varphi}(x,y) + \bar{\varphi}(\pi - x, y) = 2\,\bar{\varphi}(\tfrac{1}{2}\pi, y) + O(r^4) + O(e^{-b}) .$$

8. Approximation for large Ω

In this case the parameters s and θ are used. We shall only consider the elevation at the South coast y=0. For the elevation at the middle $(\frac{1}{2}\pi,0)$ we find from (4.11) and (4.12) the result

$$(8.1) \qquad \bar{\zeta}(\tfrac{1}{2}\pi,0) = B_o + \sum_{n=1}^{\infty} n^{-1}A_n(\sin nx+\theta_n\cos nx) + O(e^{-b}).$$

Substituting the approximations (5.12) and omitting from now on the order term with exp-b we obtain

$$(8.2) \qquad \bar{\zeta}(\tfrac{1}{2}\pi,0) = B_o\left\{1 + \sum\nolimits_1 n^{-1}ss_n \sin \tfrac{1}{2}n\pi\right\} + O(\theta),$$

or summing the series

$$(8.3) \qquad \bar{\zeta}(\tfrac{1}{2}\pi,0) = B_o \operatorname{ch} \tfrac{1}{2}s\pi + O(\theta).$$

The elevation at other points of the South coast can be found by using the first equation of (2.9) which now reduces to

$$\bar{\zeta}_x = -(p+\lambda)\bar{u},$$

so that with (4.9) and (4.14) it follows that

$$\bar{\zeta}_x = -\theta \sum\nolimits_1 \frac{n^2+s^2}{n\nu_n} A_n\sin nx + O(\theta) =$$

$$(8.4) \qquad = -\frac{4}{\pi}\theta B_o s^2\operatorname{ch}\tfrac{1}{2}s\pi \sum\nolimits_1 \frac{\sin nx}{n\nu_n} + O(\theta).$$

If this is combined with (8.3) we find

$$(8.5) \qquad \bar{\zeta}(x,0) = B_o\operatorname{ch}\tfrac{1}{2}s\pi\left\{1 + \frac{4}{\pi}\theta s^2 \sum\nolimits_1 \frac{\cos nx}{\nu_n}\right\} + O(\theta).$$

This result shows that also in this type of approximation the elevation is approximately skew symmetric with respect to the middle $x=\frac{1}{2}\pi$.

The lowest order approximation of $\bar{\zeta}(x,0)$ follows from (6.36) which gives

$$(8.6) \qquad \bar{\zeta}(x,0) = \frac{\operatorname{sh}(\frac{1}{2}s\pi+qb)-\operatorname{sh}\frac{1}{2}s\pi}{\theta s \operatorname{ch}(\frac{1}{2}s\pi+qb)} + O(1).$$

Hence in the lowest approximation the elevation at the South coast does not depend on x.

The next higher approximation needs a better approximation of B_o than that given in (6.36).

From the system (5.7) it follows that

$$(8.7) \qquad A_o = \frac{1}{\theta s \operatorname{ch}\frac{1}{2}s\pi} - \theta B_o s^4\operatorname{cth}\tfrac{1}{2}s\pi \sum\nolimits_1 \frac{\Gamma_{on}}{n\nu_n^3} + O(\theta).$$

Next the second-order approximation of the relation (6.17) gives

(8.8) $$\int_0^{\pi}(\pi\delta(x)+\theta\,\chi(x))\Big[A_0\mathrm{sh}\{s(\tfrac{1}{2}\pi-x)+qb\}-B_0\mathrm{ch}\{s(\tfrac{1}{2}\pi-x)+qb\}\,,+ - \frac{\mathrm{sh}\ s(\frac{1}{2}\pi-x)}{\theta s\ \mathrm{ch}\frac{1}{2}s\pi}\Big]dx = 0.$$

Putting by way of abbreviation

(8.9) $$A_0 = \frac{1}{\theta s\ \mathrm{ch}\frac{1}{2}s\pi} + \alpha + O(\theta)$$

(8.10) $$B_0 = \frac{\mathrm{sh}(\frac{1}{2}s\pi+qb)-\mathrm{sh}\frac{1}{2}s\pi}{\theta s\ \mathrm{ch}(\frac{1}{2}s\pi+qb)\mathrm{ch}\frac{1}{2}s\pi} + \beta + O(\theta)$$

the relation (8.8) reduces to (cf. 6.37)

(8.11) $$\frac{\mathrm{ch}\ qb-1}{s\ \mathrm{ch}(\frac{1}{2}s\pi+qb)\mathrm{ch}\frac{1}{2}s\pi}\quad\frac{1}{\pi}\int_0^{\pi}\chi(x)\mathrm{sh}\ sx\ dx + \alpha\ \mathrm{sh}(\tfrac{1}{2}s\pi+qb) + - \beta\ \mathrm{ch}(\tfrac{1}{2}s\pi+qb) + O(\theta) = 0.$$

From (6.31) it follows that

(8.12) $$\frac{1}{\pi}\int_0^{\pi}\chi(x)\mathrm{sh}\ sx\ dx = \frac{2}{\pi}\ \mathrm{sh}\ s\pi\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{\nu_n^3}$$

so that by substitution of this and the value of α from (8.7) in (8.11) we obtain

(8.13) $$\beta\ \mathrm{ch}^2(\tfrac{1}{2}s\pi+qb) = (\mathrm{ch}\ qb-1)S_1 - s^2\mathrm{sh}(\tfrac{1}{2}s\pi+qb)\{\mathrm{sh}(\tfrac{1}{2}s\pi+qb)-\mathrm{sh}\tfrac{1}{2}s\pi\}S_2$$

where

(8.14) $$S_1 = \frac{4}{\pi s}\ \mathrm{sh}\tfrac{1}{2}s\pi\ \sum_{n=1}^{\infty}(-1)^{n-1}\ \frac{n^2}{\nu_n^3}\ ,$$

and

(8.15) $$S_2 = \frac{2s}{\pi}\ \frac{1}{\mathrm{sh}\frac{1}{2}s\pi}\ \sum\nolimits_1 \frac{1}{n^2\nu_n^3}\ .$$

In order to facilitate future reference the approximate expression (8.6) for $\bar{\zeta}(x,0)$ will be denoted by $\bar{Z}(p)$ viz.

(8.16) $$\bar{Z}(p)\ \stackrel{\mathrm{def}}{=}\ \frac{\mathrm{sh}(\frac{1}{2}s\pi+qb)-\mathrm{sh}\frac{1}{2}s\pi}{q\ \mathrm{ch}(\frac{1}{2}s\pi+qb)}\ .$$

We note that this approximation gives the correct analytic result if $\Omega=0$ (cf. 7.2). The approximation holds in particular if $p\to 0$.

So far we have considered the momentary disturbance

(8.17) $$U = 0 \qquad V = -\delta(t).$$

If now the delta-function is replaced by an arbitrary time function $V = V(t)$ the Laplace transform of the elevation at the South coast can be approximated by

(8.18) $$\bar{\zeta}(x,0) \approx -\bar{V}(p)\bar{Z}(p).$$

The original may be found e.g. by the complex inversion formula

(8.19) $$\varphi(x,0,t) \approx -\frac{1}{2\pi i}\int_L e^{pt}\,\overline{V}(p)\overline{Z}(p)\,dp,$$

where L is a suitable vertical path. The right-hand side of (8.19) can be evaluated by means of the calculus of residues. There are poles of $\overline{V}(p)$ depending of course on the type of windfield and poles of $\overline{Z}(p)$. The complex function $\overline{Z}(p)$ is regular at the origin and

(8.20) $$\overline{Z}(0) = b.$$

Further there is a set of poles determined by ch $\beta = 0$ or

(8.21) $$s(\tfrac{1}{2}\pi+\Theta b) = \pm\,(m+\tfrac{1}{2})\,\pi i, \qquad m=0,1,2\ldots$$

For each value of m we obtain three poles, one real and negative tying in the interval $(-\lambda,0)$ and two conjugate complex ones with a negative real part.

Qualitatively these poles correspond to the eigenvalues of the problem. However, the deviations between corresponding poles and eigenvalues will be the greater according as the approximation (8.16) is more inaccurate. The lowest real pole of $\overline{Z}(p)$ which is obtained from (8.21) for m=0 may be expected to give a very good approximation to the lowest real eigenvalue of the problem. This point will be taken up later on in the numerical applications.

The approximation (8.16) may be interpreted in the following interesting way. If we expand $\overline{Z}(p)$ in powers of $e^{-\beta}$

(8.22) $$\overline{Z}(p) = q^{-1}\left\{1-(e^{s\pi}-1)e^{-\beta}-2e^{-2\beta}+(e^{s\pi}-1)e^{-3\beta}+\ldots\right\},$$

the first term on the right-hand side represents the direct disturbance at the coast y=0. The second term represents the influence of the disturbance from the ocean arising after b dimensionless time-units. The third term is the reflection of the disturbance at the coast with respect to the ocean. The latter disturbance begins to act after 2b time-units etcetera.

Finally we give an expression in the frictionless case $\lambda=0$. Then

(8.23) $$\overline{Z}(p) = p^{-1}\left\{\mathrm{th}(\tfrac{1}{2}\Omega\pi+pb)-\mathrm{sech}(\tfrac{1}{2}\Omega\pi+pb)\,\mathrm{sh}\tfrac{1}{2}\Omega\pi\right\}.$$

Its original can be written down at once viz.

(8.24) $$Z(t) = H(t) - 2\mathrm{sh}\tfrac{1}{2}\Omega\pi\, H(t-b)-2e^{-\Omega\pi}H(t-2b)+$$
$$+\, 2\mathrm{sh}\tfrac{1}{2}\Omega\pi\, H(t-3b)+\ldots,$$

where H(t) is Heaviside's unit-function

(8.25) $$H(t) \overset{\mathrm{def}}{=} \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0. \end{cases}$$

References [1]

H.A. Lauwerier and D. van Dantzig [1]. General considerations concerning the hydrodynamical problem of the motion of the North Sea. North Sea Problem I. Proc.K.A.v.W. A 63. Ind.Math.22, 170-180 (1960).

H.A. Lauwerier [2]. On certain trigonometrical expansions. J. of Math. and Mech. Vol.8, 419-432 (1959).

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1) Cf. also the bibliography of the preceding papers and in particular that of North Sea Problem I.